Received: *April 7, 2000*, Accepted: *May 24, 2000*

Perturbative beta function of $N = 2$ super Yang-Mills theories

Alberto Blasi[a], Vitor E.R. Lemes[b], Nicola Maggiore[a], Silvio P. Sorella[b], Alessandro Tanzini[c], Ozemar S. Ventura[b] and Luiz C.Q. Vilar[b]

[a] *Dipartimento di Fisica, Università di Genova and I.N.F.N., Sezione di Genova via Dodecaneso 33, I-16146 Genova, Italy*
[b] *UERJ, Universidade do Estado do Rio de Janeiro, Departamento de Física Teórica Rua São Francisco Xavier, 524, 20550-013, Maracanã, Rio de Janeiro, Brazil*
[c] *Dipartimento di Fisica, Università di Roma "Tor Vergata" via della Ricerca Scientifica 1, I-00133 Roma, Italy*
E-mail: blasi@ge.infn.it, vitor@dft.if.uerj.br, maggiore@ge.infn.it, sorella@dft.if.uerj.br, tanzini@roma2.infn.it, ozemar@dft.if.uerj.br, lcqvilar@dft.if.uerj.br

Abstract: An algebraic proof of the nonrenormalization theorem for the perturbative beta function of the coupling constant of $N = 2$ Super Yang-Mills theory is provided. The proof relies on a fundamental relationship between the $N = 2$ Yang-Mills action and the local gauge invariant polynomial $\mathrm{Tr}\,\phi^2$, $\phi(x)$ being the scalar field of the $N = 2$ vector gauge multiplet. The nonrenormalization theorem for the β_g function follows from the vanishing of the anomalous dimension of $\mathrm{Tr}\,\phi^2$.

Keywords: BRST Quantization, Topological Field Theories, BRST Symmetry, Extended Supersymmetry.

JHEP05(2000)039

Contents

1. Introduction

It has long been known that $N = 2$ Super Yang-Mills (SYM) theory displays a set of far-reaching and remarkable features, both at the nonperturbative and perturbative level. For instance, effects due to instantons can be taken into account in an exact way [1, 2]. Needless to say, the $N = 2$ Yang-Mills theory is the corner stone of the duality mechanism discussed by Seiberg and Witten [3] in order to characterize the vacuum structure of various supersymmetric gauge models.

Concerning now the perturbative regime, certainly the most celebrated result is the nonrenormalization theorem of the beta function of the coupling constant g (β_g), stating that β_g receives only one-loop contributions. In other words, if the β_g function vanishes at the one-loop order, it will vanish at all orders of the perturbation theory. This unique behavior is usually understood in terms of the analogous Adler-Bardeen theorem for the U(1) axial current which, due to supersymmetry, belongs to the same supercurrent multiplet of the energy-momentum tensor. As a consequence, a deep and strong relationship between the β_g function and the coefficient of the axial anomaly is expected to hold. However, up to our knowledge, an algebraic proof of the nonrenormalization theorem of the β_g function of $N = 2$, based on Ward identities and on BRS invariance, is still lacking. Indeed the arguments which lead to the exactness of the one-loop result for the β-function are based either on a regularization procedure which does not respect the supersymmetry or on the higher derivative method which has not really been implemented for the $N = 2$ case. A comprehensive account of the situation is contained in ref. [4], whose point of view we would like to quote (ibid. pag 195): "Here we have stressed these weaknesses not because of a mistrust in the arguments for finiteness, but to show that they are not proofs in a mathematical sense and that there is still room for further work". The aim of the present work is to fill this gap, providing a purely algebraic proof of the aforementioned theorem.

Before entering into the technical aspects of the paper, it is worth spending a few words on the strategy of our proof. First of all, instead of working with the spinor indices of supersymmetry $(\alpha, \dot{\alpha})$, we shall make use of the well known twisting procedure [5], allowing us to replace the indices $(\alpha, \dot{\alpha})$ with Lorentz vector indices. Of course, the physical content of the theory will be left unchanged, for the twist is a linear change of variables, and a twisted version of the model is perturbatively indistinguishable from the original one. However, the use of the twisted variables has the great advantage of considerably simplifying the full computation of the relevant BRS cohomology classes. In particular, it has been possible to prove [6] that the action of the $N = 2$ Yang-Mills can be related to the local gauge invariant polynomial $\mathrm{Tr}\,\phi^2$, $\phi(x)$ being the scalar field of the $N = 2$ multiplet. This important relation will be the essential ingredient for the proof of the nonrenormalization theorem, as the operator $\mathrm{Tr}\,\phi^2$ possesses remarkable ultraviolet finiteness properties.

Another relevant feature of the twisted formulation is the appearance of a scalar supersymmetry charge δ. In fact, the Ward identities associated to the scalar supersymmetry transformations play a crucial role in proving that the anomalous dimension of $\mathrm{Tr}\,\phi^2$ is actually vanishing. Thus, making use of the relationship between $\mathrm{Tr}\,\phi^2$ and the $N = 2$ action, we shall succeed in promoting the ultraviolet finiteness properties of $\mathrm{Tr}\,\phi^2$ to the β_g function of $N = 2$, proving therefore its nonrenormalization theorem.

It is worth mentioning that the twisting procedure has recently been successfully employed to analyze nonperturbative effects, as it allows to understand the $N = 2$ Yang-Mills theory in terms of the topological Witten's gauge theory [5], providing deep insights for the nonperturbative regime of the model [7]. In particular, it has been shown that the remarkable nonrenormalization properties of the holomorphic part of the effective action of the $N = 2$ SYM [8] are ensured in the context of instanton calculus by the Ward identities associated to the scalar supersymmetry transformations. It is interesting to remark that the nonrenormalization properties both of the perturbative and the nonperturbative sector of the $N = 2$ SYM theory can be related to the Ward identities associated to the scalar supersymmetry present in its twisted formulation.

JHEP05(2000)039

The organization of the paper is as follows. Section 2 is devoted to a brief review of the twisting procedure for $N = 2$ supersymmetry. In section 3 the quantization of the theory in the Wess-Zumino gauge is performed following the BRS procedure [6, 19, 20]. In section 4 the relationship between the gauge invariant polynomial $\mathrm{Tr}\,\phi^2$ and the $N = 2$ Yang-Mills action will be presented. Section 5 will deal with the ultraviolet finiteness properties of $\mathrm{Tr}\,\phi^2$. Finally, in section 6 the algebraic proof of the nonrenormalization theorem for the β_g function of $N = 2$ Yang-Mills will be given.

2. The twist

The starting key of addressing the problem is to work with the twisted version of $N = 2$ SYM theories, which, in the Wess-Zumino (WZ) gauge, coincide with Topological Yang-Mills (TYM) theory. In this section, we sketch the basics of the twisting procedure, originally introduced by Witten in [5] and fully developed for $N = 2$ SYM in [6, 9].

The $N = 2$ susy algebra reads

$$
\begin{aligned}
\{\mathcal{Q}^i_\alpha, \overline{\mathcal{Q}}_{j\dot\alpha}\} &= \delta^i_j(\sigma^\mu)_{\alpha\dot\alpha}\partial_\mu \\
\{\mathcal{Q}^i_\alpha, \mathcal{Q}^j_\beta\} &= \{\overline{\mathcal{Q}}_{i\dot\alpha}, \overline{\mathcal{Q}}_{j\dot\beta}\} = 0\,,
\end{aligned}
\tag{2.1}
$$

where $(\mathcal{Q}^i_\alpha, \overline{\mathcal{Q}}_{j\dot\alpha})$ are the supersymmetry charges, indexed by $i = 1, 2$ and Weyl spinor indices $\alpha, \dot\alpha = 1, 2$. The total number of supercharges is therefore eight. The simple observation that the indices (i, α) both run from one to two, suggests the idea of identifying the supersymmetry index with the spinor index

$$
i \equiv \alpha\,,
\tag{2.2}
$$

operation which amounts to a modification of the rotation group of the theory

$$
\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \longrightarrow \mathrm{SU}(2)_L \times \mathrm{SU}(2)'_R\,,
\tag{2.3}
$$

where $\mathrm{SU}(2)'_R$ is the diagonal sum of the original $\mathrm{SU}(2)_R$ and $\mathrm{SU}(2)_I$, the group of transformations of the supersymmetry index i. The global symmetry group of $N = 2$ SYM finally is

$$
\mathrm{SU}(2)_L \times \mathrm{SU}(2)'_R \times \mathrm{U}(1)_R\,,
\tag{2.4}
$$

where $\mathrm{U}(1)_R$ is associated to the $\mathcal{R}$-symmetry, according to which the charges $\mathcal{Q}^i_\alpha$ and $\overline{\mathcal{Q}}_{i\dot\alpha}$ are assigned eigenvalues $+1$ and -1 respectively.

The twisted supercharges are

$$
\begin{aligned}
\mathcal{Q}^i_\alpha &\longrightarrow \mathcal{Q}^\beta_\alpha \\
\overline{\mathcal{Q}}_{i\dot\alpha} &\longrightarrow \overline{\mathcal{Q}}_{\alpha\dot\alpha}\,.
\end{aligned}
\tag{2.5}
$$

JHEP05(2000)039

The procedure can then be pushed further to the point of getting rid of the spinor index. Indeed, $\mathcal{Q}^{\beta}_{\alpha}$ and $\overline{\mathcal{Q}}_{\alpha\dot\alpha}$ can be rearranged as follows

$$\delta \equiv \frac{1}{\sqrt{2}}\varepsilon^{\alpha\beta}\mathcal{Q}_{\alpha\beta} \tag{2.6}$$

$$\delta_\mu \equiv \frac{1}{\sqrt{2}}\overline{\mathcal{Q}}_{\alpha\dot\alpha}(\sigma^\mu)^{\dot\alpha\alpha} \tag{2.7}$$

$$\delta_{\mu\nu} \equiv \frac{1}{\sqrt{2}}(\sigma_{\mu\nu})^{\alpha\beta}\mathcal{Q}_{\beta\alpha}\,, \tag{2.8}$$

where we adopted the usual conventions for the quantities $\varepsilon^{\alpha\beta}$, $(\sigma^\mu)^{\dot\alpha\alpha}$ and $(\sigma_{\mu\nu})^{\alpha\beta}$ [10]. Notice that, due to the properties of $\sigma_{\mu\nu}$, the operator $\delta_{\mu\nu}$ is selfdual:

$$\delta_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\delta^{\rho\sigma}\,, \tag{2.9}$$

reducing to three the number of its independent components.

The familiar supersymmetry algebra (2.1) correspondingly twists into

$$\begin{aligned} \delta^2 &= 0 \\ \{\delta,\delta_\mu\} &= \partial_\mu \\ \{\delta_\mu,\delta_\nu\} &= 0 \end{aligned} \tag{2.10}$$

$$\begin{aligned} \{\delta_\mu,\delta_{\rho\sigma}\} &= -(\varepsilon_{\mu\nu\rho\sigma}\partial^\nu + g_{\mu\rho}\partial_\sigma - g_{\mu\sigma}\partial_\rho) \\ \{\delta_{\mu\nu},\delta\} &= \{\delta_{\mu\nu},\delta_{\rho\sigma}\} = 0 \end{aligned} \tag{2.11}$$

A key feature of the twisted algebra is the appearance of the scalar supersymmetry charge δ, which is still an invariance of the theory when this is formulated on a generic (differential) manifold M. We recall in fact that in Witten's Topological Yang-Mills theory the observables are *defined* as cohomology classes of δ, which is thus treated in this context as a BRS-like operator.[1] The corresponding correlation functions are non vanishing in the nonperturbative sector of the theory and can can be indeed evaluated by a semiclassical expansion around an instanton background. They are independent of the metric on M by virtue of the Ward identities associated to δ, and are related with the Donaldson invariants [5]. The underlying topological nature of the twisted theory is already revealed by the subalgebra (2.10), which allows to write the space-time derivative as a δ-exact term. Notice that this subalgebra is a common feature of all known topological quantum field theories [11, 12].

The price of working in the Wess-Zumino gauge, is the appearance in the r.h.s. of (2.1) (and hence of (2.10) and (2.11)) of breakings, which take the form of field dependent gauge transformations and equations of motion. The drawbacks deriving from those breakings for the quantization of the theory have been stressed by Breitenlohner and Maison in [13].

[1] Actually since the Witten's theory is formulated in the Wess-Zumino gauge, one is led to analyze the equivariant cohomology of the corresponding BRS operator, as observed in [23].

The fields of $N = 2$ WZ gauge multiplet $(A_\mu, \psi^i_\alpha, \overline{\psi}^i_{\dot\alpha}, \phi, \overline{\phi})$, belonging to the adjoint representation of the gauge group, after the twisting procedure become

$$(A_\mu, \psi_\mu, \chi_{\mu\nu}, \eta, \phi, \overline{\phi}) \,. \tag{2.12}$$

The twist, acting on the internal index i, leaves the gauge connection A_μ and the scalars $(\phi, \overline{\phi})$ unaltered, while the spinors $(\psi^i_\alpha, \overline{\psi}^i_{\dot\alpha})$ turn into

$$\begin{aligned} \psi^i_\beta &\to = \frac{1}{2}(\psi_{(\alpha\beta)} + \psi_{[\alpha\beta]}) \\ \overline{\psi}^i_{\dot\alpha} &\to \overline{\psi}_{\alpha\dot\alpha} \to \psi_\mu = (\overline{\sigma}_\mu)^{\alpha\dot\alpha}\overline{\psi}_{\alpha\dot\alpha} \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} \psi_{(\alpha\beta)} &\to \chi_{\mu\nu} = (\sigma_{\mu\nu})^{\alpha\beta}\psi_{(\alpha\beta)} \\ \psi_{[\alpha\beta]} &\to \eta = \varepsilon^{\alpha\beta}\psi_{[\alpha\beta]} \,. \end{aligned} \tag{2.14}$$

Notice that $(\psi_\mu, \chi_{\mu\nu}, \eta)$ anticommute, due to their spinor origin.

Finally, twisting the action of $N = 2$ SYM, remarkably yields the TYM action [5]

$$\mathcal{S}^{N=2}_{SYM}(A_\mu, \psi^i_\alpha, \overline{\psi}^i_{\dot\alpha}, \phi, \overline{\phi}) \longrightarrow \mathcal{S}_{TYM}(A_\mu, \psi_\mu, \chi_{\mu\nu}, \eta, \phi, \overline{\phi}) \tag{2.15}$$

and

$$\begin{aligned} \mathcal{S}_{TYM} = \frac{1}{g^2}\,\mathrm{Tr}\int d^4x\, \Big(& \frac{1}{2}F^+_{\mu\nu}F^{+\mu\nu} - \chi^{\mu\nu}(D_\mu\psi_\nu - D_\nu\psi_\mu)^+ + \\ & + \eta D_\mu\psi^\mu - \frac{1}{2}\overline{\phi}D_\mu D^\mu\phi + \frac{1}{2}\overline{\phi}\,\{\psi^\mu, \psi_\mu\} - \\ & - \frac{1}{2}\phi\,\{\chi^{\mu\nu}, \chi_{\mu\nu}\} - \frac{1}{8}\,[\phi, \eta]\,\eta - \frac{1}{32}\,[\phi, \overline{\phi}]\,[\phi, \overline{\phi}]\Big) \,, \end{aligned} \tag{2.16}$$

where

$$\begin{aligned} F^+_{\mu\nu} &= F_{\mu\nu} + \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} \,, \\ \widetilde{F}^+_{\mu\nu} &= \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{+\rho\sigma} = F^+_{\mu\nu} \,, \\ (D_\mu\psi_\nu - D_\nu\psi_\mu)^+ &= (D_\mu\psi_\nu - D_\nu\psi_\mu) + \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}(D^\rho\psi^\sigma - D^\sigma\psi^\rho) \,. \end{aligned} \tag{2.17}$$

Notice that the theory has the unique coupling constant g^2, which in this parametrization is easily related to the number L of loops in the Feynman diagrams computations:

$$L \text{ loops} \longleftrightarrow (g^2)^{L-1} \,. \tag{2.18}$$

The action $\mathcal{S}_{TYM}$ (2.16) is left invariant by the twisted supersymmetry charges

$$\delta\,\mathcal{S}_{TYM} = \delta_\mu\,\mathcal{S}_{TYM} = \delta_{\mu\nu}\,\mathcal{S}_{TYM} = 0 \,, \tag{2.19}$$

and it is also gauge invariant, i.e.

$$\delta_{\text{gauge}}\,\mathcal{S}_{TYM} = 0 \,. \tag{2.20}$$

We stress that the twist simply corresponds, on the flat euclidean spacetime $\mathbb{R}^4$, to a linear change of variables, and therefore the twisted theory is perturbatively indistinguishable from the original one. In particular, all the perturbative results are the same for the two theories. This fact has been already verified under many circumstances: we just recall here the *algebraic* absence of gauge anomalies,[2] which is quite uncommon for four dimensional gauge field theories. Indeed, in these cases it usually happens that the gauge anomaly is algebraically allowed, and only the vanishing of its coefficient may be invoked to recover perturbative renormalization. Another quantity which has been verified to go over unchanged to the twisted related theories is the value of the 1-loop β function of the coupling g, which is universal, i.e. scheme-independent.[3] What is still missing, is the main aim of the present paper, namely the algebraic, regularization independent proof of the celebrated vanishing above 1 loop of the β_g function of $N = 2$ SYM theories, and hence of TYM theory. This means, according to the parametrization adopted for the action $\mathcal{S}_{TYM}$ (2.16), that β_g is proportional to g^3, to all orders of perturbation theory

$$\beta_g \propto g^3 \,. \tag{2.21}$$

We will be able in the next sections to precise our claim (2.21), which makes sense only after a particular renormalization scheme has been adopted, since it is evident that a change of scheme generally induces in the 1 loop only β function terms of arbitrarily high power of the coupling constant. We address the interested reader to the evergreen lectures given by D.J. Gross [18].

3. Quantum extension

For what follows, it is useful to briefly illustrate the technique employed for the quantum extension of the model. As already said, the way in which the algebraic structure is broken in the Wess-Zumino gauge, is incompatible with the construction of a quantum vertex Γ satisfying quantum symmetries (gauge and supersymmetry) [13]. The puzzle has been solved in [14, 19, 20], where the renormalization of $N = 2$ SYM has been performed, and the same procedure has been successfully applied in [6] for TYM. The idea is to collect all the relevant symmetries of the theory into an unique operator, by means of some global ghosts. Here, by relevant symmetries we mean the ordinary BRS symmetry s, and the symmetries of the topological subalgebra (2.10), the charges $\delta_{\mu\nu}$ being redundant in order to define the theory [6]. Let us therefore define the extended BRS operator

$$\mathcal{Q} = s + \omega\delta + \varepsilon^\mu \delta_\mu \,, \tag{3.1}$$

[2] The algebraic absence of anomalies has been proven in [6] for TYM and in [14] for $N = 2$ SYM.

[3] The 1 loop β function has been calculated in [15] for TYM and in [16, 17] for $N = 2$ SYM

where ω and ε^μ are global ghosts. The action of $\mathcal{Q}$ on the fields belonging to the twisted $N=2$ gauge supermultiplet is

$$
\begin{aligned}
\mathcal{Q}A_\mu &= -D_\mu c + \omega\psi_\mu + \frac{\varepsilon^\nu}{2}\chi_{\nu\mu} + \frac{\varepsilon_\mu}{8}\eta \\
\mathcal{Q}\psi_\mu &= \{c,\psi_\mu\} - \omega D_\mu\phi + \varepsilon^\nu\left(F_{\nu\mu} - \frac{1}{2}F^+_{\nu\mu}\right) - \frac{\varepsilon_\mu}{16}[\phi,\overline{\phi}] \\
\mathcal{Q}\chi_{\sigma\tau} &= \{c,\chi_{\sigma\tau}\} + \omega F^+_{\sigma\tau} + \frac{\varepsilon^\mu}{8}(\varepsilon_{\mu\sigma\tau\nu} + g_{\mu\sigma}g_{\nu\tau} - g_{\mu\tau}g_{\nu\sigma})D^\nu\overline{\phi} \\
\mathcal{Q}\eta &= \{c,\eta\} + \frac{\omega}{2}[\phi,\overline{\phi}] + \frac{\varepsilon^\mu}{2}D_\mu\overline{\phi} \\
\mathcal{Q}\phi &= [c,\phi] - \varepsilon^\mu\psi_\mu \\
\mathcal{Q}\overline{\phi} &= \left[c,\overline{\phi}\right] + 2\omega\eta\,.
\end{aligned}
\tag{3.2}
$$

Moreover, $\mathcal{Q}$ is defined on the $\Phi\Pi$ - multiplet ghost-antighost-Lagrange multiplier $(c,\bar{c},b)$ as follows:

$$
\begin{aligned}
\mathcal{Q}c &= c^2 - \omega^2\phi - \omega\varepsilon^\mu A_\mu + \frac{\varepsilon^2}{16}\overline{\phi} \\
\mathcal{Q}\overline{c} &= b \\
\mathcal{Q}b &= \omega\varepsilon^\mu\partial_\mu\overline{c}\,.
\end{aligned}
\tag{3.3}
$$

Besides describing a symmetry of the action $\mathcal{S}_{TYM}$, the relevant property of the operator $\mathcal{Q}$ is:

$$
\mathcal{Q}^2 = \omega\varepsilon^\mu\partial_\mu + \text{ eqs. of motion}\,,
\tag{3.4}
$$

which allows for a standard application of the BRS technique, since the extended BRS operator $\mathcal{Q}$ is on-shell nilpotent in the space of integrated local functionals.

The complete classical action is [6]

$$
\Sigma = \mathcal{S}_{TYM} + S_{gf} + S_{ext} + S_\Box\,,
\tag{3.5}
$$

where

$$
S_{gf} = \mathcal{Q}\int d^4x\ \mathrm{Tr}(\overline{c}\partial A)
\tag{3.6}
$$

is the (Landau) gauge fixing term;

$$
\mathcal{S}_{\rm ext} = \mathrm{Tr}\int d^4x\left(L\mathcal{Q}c + D\mathcal{Q}\phi + \Omega^\mu\mathcal{Q}A_\mu + \xi^\mu\mathcal{Q}\psi_\mu + \rho\mathcal{Q}\overline{\phi} + \tau\mathcal{Q}\eta + \frac{1}{2}B^{\mu\nu}\mathcal{Q}\chi_{\mu\nu}\right),
\tag{3.7}
$$

couples external sources (called antifields in the Batalin-Vilkovisky formalism) $(L,D,\Omega^\mu,\xi^\mu,\rho,\tau,B_{\mu\nu})$, to the nonlinear variations of the quantum fields $(c,\phi,A_\mu,$ $\psi_\mu,\overline{\phi},\eta,\chi_{\mu\nu})$ respectively; finally

$$
\mathcal{S}_\Box = \frac{g^2}{4}\mathrm{Tr}\int d^4x\left(\frac{1}{2}\omega^2B^{\mu\nu}B_{\mu\nu} - \omega B^{\mu\nu}\varepsilon_\mu\xi_\nu - \frac{1}{8}\varepsilon^\mu\varepsilon^\nu\xi_\mu\xi_\nu + \frac{1}{8}\varepsilon^2\xi^2\right),
\tag{3.8}
$$

is introduced in order to take care of the fact that $\mathcal{Q}$ is nilpotent (on integrated functionals) once the equations of motion are used.

JHEP05(2000)039

The action Σ (3.5) satisfies an extended Slavnov-Taylor (ST) identity

$$\mathcal{S}(\Sigma) = \omega\varepsilon^\mu \Delta^{cl}_\mu \,, \tag{3.9}$$

where

$$\begin{aligned}\mathcal{S}(\Sigma) = \mathrm{Tr} \int d^4x \Bigg(& \frac{\delta\Sigma}{\delta A^\mu}\frac{\delta\Sigma}{\delta\Omega_\mu} + \frac{\delta\Sigma}{\delta\xi^\mu}\frac{\delta\Sigma}{\delta\psi_\mu} + \frac{\delta\Sigma}{\delta L}\frac{\delta\Sigma}{\delta c} + \frac{\delta\Sigma}{\delta D}\frac{\delta\Sigma}{\delta\phi} + \frac{\delta\Sigma}{\delta\rho}\frac{\delta\Sigma}{\delta\overline{\phi}} + \\ & + \frac{\delta\Sigma}{\delta\tau}\frac{\delta\Sigma}{\delta\eta} + \frac{1}{2}\frac{\delta\Sigma}{\delta B^{\mu\nu}}\frac{\delta\Sigma}{\delta\chi_{\mu\nu}} + b\frac{\delta\Sigma}{\delta\overline{c}} + \omega\varepsilon^\mu\partial_\mu\overline{c}\frac{\delta\Sigma}{\delta b}\Bigg) \,.\end{aligned} \tag{3.10}$$

In the above expression (3.9) for the extended ST identity, Δ^{cl}_μ is an integrated linear polynomial in the quantum fields:

$$\Delta^{cl}_\mu = \mathrm{Tr}\int d^4x \left(L\partial_\mu c - D\partial_\mu\phi - \Omega^\nu\partial_\mu A_\nu + \xi^\nu\partial_\mu\psi_\nu \ - \rho\partial_\mu\overline{\phi} + \tau\partial_\mu\eta + \frac{1}{2}B^{\nu\sigma}\partial_\mu\chi_{\nu\sigma} \right) ; \tag{3.11}$$

therefore, Δ^{cl}_μ does not get quantum corrections and the breaking of the ST identity (3.9) is only *classical*.

The whole algebra is summarized into

$$\mathcal{B}_\Sigma\mathcal{B}_\Sigma = \omega\varepsilon^\mu\mathcal{P}_\mu \,, \tag{3.12}$$

where

$$\begin{aligned}\mathcal{B}_\Sigma = \mathrm{Tr}\int d^4x \Bigg(& \frac{\delta\Sigma}{\delta A^\mu}\frac{\delta}{\delta\Omega_\mu} + \frac{\delta\Sigma}{\delta\Omega_\mu}\frac{\delta}{\delta A^\mu} + \frac{\delta\Sigma}{\delta\psi_\mu}\frac{\delta}{\delta\xi^\mu} + \frac{\delta\Sigma}{\delta\xi^\mu}\frac{\delta}{\delta\psi_\mu} + \frac{\delta\Sigma}{\delta L}\frac{\delta}{\delta c}\Bigg) + \\ & + \frac{\delta\Sigma}{\delta c}\frac{\delta}{\delta L} + \frac{\delta\Sigma}{\delta\phi}\frac{\delta}{\delta D} + \frac{\delta\Sigma}{\delta D}\frac{\delta}{\delta\phi} + \frac{\delta\Sigma}{\delta\overline{\phi}}\frac{\delta}{\delta\rho} + \frac{\delta\Sigma}{\delta\rho}\frac{\delta}{\delta\overline{\phi}} + \frac{\delta\Sigma}{\delta\eta}\frac{\delta}{\delta\tau} + \\ & + \frac{\delta\Sigma}{\delta\tau}\frac{\delta}{\delta\eta} + \frac{1}{2}\frac{\delta\Sigma}{\delta\chi_{\mu\nu}}\frac{\delta}{\delta B^{\mu\nu}} + \frac{1}{2}\frac{\delta\Sigma}{\delta B^{\mu\nu}}\frac{\delta}{\delta\chi_{\mu\nu}} + b\frac{\delta}{\delta\overline{c}} + \omega\varepsilon^\mu\partial_\mu\overline{c}\frac{\delta}{\delta b}\Bigg) \,,\end{aligned} \tag{3.13}$$

is the linearized extended ST operator, and

$$\begin{aligned}\mathcal{P}_\mu &= \sum\nolimits_i \int d^4x \left(\partial_\mu\varphi^i\frac{\delta}{\delta\varphi^i} + \partial_\mu\varphi^{*i}\frac{\delta}{\delta\varphi^{*i}} \right) = 0 \,, \\ \varphi^i &= \text{all the fields } \left(A, \psi, \phi, \overline{\phi}, \eta, \chi, c, \overline{c}, b\right) , \\ \varphi^{*i} &= \text{all the antifields } (\Omega, \xi, L, D, \rho, \tau, B) \,.\end{aligned} \tag{3.14}$$

is the functional generator of spacetime translations.

In [6], the renormalization program has been completed by showing that the action Σ is stable under radiative corrections and that no anomalies occur, so that the classical relation (3.9) can safely be implemented for the 1PI generating functional

$$\Gamma = \Sigma + O(\hbar) \,, \tag{3.15}$$

i.e.

$$\mathcal{S}(\Gamma) = \omega\varepsilon^\mu\Delta^{cl}_\mu \,. \tag{3.16}$$

Let us close this section by summarizing the quantum numbers of the fields and parameters entering the theory in tables 1, 2 and 3.

	A_μ	$\chi_{\mu\nu}$	ψ_μ	η	ϕ	$\overline{\phi}$
dim.	1	3/2	3/2	3/2	1	1
$\mathcal{R}$-charge	0	-1	1	-1	2	-2
gh-number	0	0	0	0	0	0
nature	comm.	ant.	ant.	ant.	comm.	comm.

Table 1: Quantum numbers: quantum fields.

	c	$\bar{c}$	ω	ε_μ
dim.	0	2	$-1/2$	$-1/2$
$\mathcal{R}$-charge	0	0	-1	1
gh-number	1	-1	1	1
nature	ant.	ant.	comm.	comm.

Table 2: Quantum numbers: ghosts.

	L	D	γ^μ	ξ^μ	ρ	τ	$B^{\mu\nu}$
dim.	4	3	3	5/2	3	5/2	5/2
$\mathcal{R}$-charge	0	-2	0	-1	2	1	1
gh-number	-2	-1	-1	-1	-1	-1	-1
nature	comm.	ant.	ant.	comm.	ant.	comm.	comm.

Table 3: Quantum numbers: external sources.

4. The action, and its relation with $\mathrm{Tr}\,\phi^2$

By expanding the linearized ST operator (3.13) in powers of the global ghost ε^μ, we have

$$\mathcal{B}_\Sigma = b_\Sigma + \varepsilon^\mu \mathcal{W}_\mu + \frac{1}{2}\varepsilon^\mu\varepsilon^\nu \mathcal{W}_{\mu\nu}\,, \tag{4.1}$$

and the algebraic relation (3.12) implies that the operators b_Σ and $\mathcal{W}_\mu$ must obey

$$\begin{aligned} b_\Sigma b_\Sigma &= 0 \\ \{b_\Sigma, \mathcal{W}_\mu\} &= \omega \mathcal{P}_\mu\,. \end{aligned} \tag{4.2}$$

As it is well known [12, 21], a theorem assures that the integrated cohomology of $\mathcal{B}_\Sigma$ is isomorphic to a subspace of the integrated cohomology of the first term in the expansion (4.1), i.e. b_Σ, whose local cohomology is known [22] to consist of invariant polynomials in the scalar field ϕ

$$\mathcal{P}_n(\phi) = \mathrm{Tr}\left(\frac{\phi^n}{n}\right), \qquad n \geq 2\,. \tag{4.3}$$

Now, the polynomials $\mathcal{P}_n(\phi)$ become cohomologically trivial if we relax the condition of analyticity in the global ghost ω. In fact, for instance,

$$\mathcal{P}_2(\phi) = \frac{1}{2}\,\mathrm{Tr}\,\phi^2 \;= \frac{1}{2} b_\Sigma\,\mathrm{Tr}\left(-\frac{1}{\omega^2}c\phi + \frac{1}{3\omega^4}c^3\right). \tag{4.4}$$

This means that the local cohomology of the operator b_Σ, and hence the integrated cohomology of the full operator $\mathcal{B}_\Sigma$, is trivial if we take into account also field polynomials which *are not* analytic in the parameter ω. On the other way, we have to remember that the physical content of TYM theory is not empty, being given by the non empty cohomology of $\mathcal{B}_\Sigma$, with the requirement of analyticity in ω, or by the equivariant cohomology of the ordinary BRS operator, as defined in [23], where TYM is seen as a topological quantum field theory of the Witten type, i.e. generated entirely by a BRS cocycle. More on this can be found in [6]. Thus, the requirement of analyticity in ω is not just a matter of convenience, but it is founded in the quantum field theory rules. However, in perturbation theory the diagrammatic expansion of the generating functional Γ is obviously analytic in the parameters of the theory. In other words, counterterms for the action must be analytic in ω, while this is not necessarily so for other elements of the quantum theory, like for instance quantum insertions of the type $\partial\Gamma/\partial\alpha$, where α is any parameter of the theory. Indeed, the possible poles can be easily removed from $\partial\Gamma/\partial\alpha$ by a multiplication for a suitable power of ω.

Turning back to the original issue, in the space of integrated local functionals of vanishing $\mathcal{R}$-charge, zero ghost number and canonical dimension four, the cohomology of $\mathcal{B}_\Sigma$ is given by the unique element [6]

$$\varepsilon^{\mu\nu\rho\tau}\mathcal{W}_\mu\mathcal{W}_\nu\mathcal{W}_\rho\mathcal{W}_\tau \int d^4x \ \mathrm{Tr}\left(\frac{\phi^2}{2}\right), \tag{4.5}$$

where $\mathcal{W}_\mu$ is the operator defined by the filtration of $\mathcal{B}_\Sigma$ in (4.1). It is clear from (4.5) that the role of the operator $\mathcal{W}_\mu$ is that of extracting the integrated cohomology classes from the non-integrated ones, in perfect analogy with what happens in topological field theories, which are all characterized by the algebraic relation (4.2), according to which the spacetime derivative is written as an anticommutator between a BRS operator and a vectorial supersymmetry [11, 12]. The operator $\mathcal{W}_\mu$ acts therefore as a climbing up operator for the descent equations, and the non trivial part of the action is obtained by repeated applications of $\mathcal{W}_\mu$ to the lowest term of the ladder [12]. For our concern, the twisted action (3.5) of $N=2$ SYM can be rewritten, modulo a trivial $\mathcal{B}_\Sigma$-cocycle, as

$$\Sigma \approx -\frac{1}{3g^2}\mathcal{W}^4 \int d^4x \ \mathrm{Tr}\,\frac{\phi^2}{2}, \tag{4.6}$$

where

$$\mathcal{W}^4 \equiv \varepsilon^{\mu\nu\rho\sigma}\mathcal{W}_\mu\mathcal{W}_\nu\mathcal{W}_\rho\mathcal{W}_\sigma\,. \tag{4.7}$$

The relation (4.6) is of fundamental importance for what follows. Although it is self-explanatory, it is worth to stress that the bulk of the theory is the composite operator $\mathrm{Tr}\,\phi^2/2$, which somehow contains all the information on the physics of the theory. The operator $\mathcal{W}_\mu$ is nothing more than a BRS decomposition of the spacetime derivative. In this respect, $\mathrm{Tr}\,\phi^2/2$ can be regarded as a sort of prepotential of $N=2$ SYM (and of TYM, too), at least in this twisted version. Notice also that $\phi(x)$ is just the scalar field of the original, untwisted, theory, namely the scalar of the gauge multiplet of $N=2$ SYM, and that the composite operator $\mathrm{Tr}\,\phi^2/2$ is already known to play a decisive role for the exact solution of the theory. In fact, it is precisely its v.e.v. $\langle \mathrm{Tr}\,\phi^2/2\rangle = u$ which parametrizes the gauge non equivalent vacua of the theory [3]. Here, it is instructive to recover the key role of this composite operator in a rather different context. Intuitively, it is also clear what will our strategy be for the following: the nonrenormalization properties of the full action are closely related to those of its "generating kernel" $\mathrm{Tr}\,\phi^2/2$, and on this issue we will concentrate in the next sections.

5. Non renormalization of $\mathrm{Tr}\,\phi^2$

In order to show that the composite operator $\mathrm{Tr}\,\phi^2/2$ has vanishing anomalous dimensions, the ordinary BRS and δ-symmetry (2.6) are sufficient. Therefore, from now on throughout the rest of this paper, we shall put $\varepsilon^\mu \equiv 0$, and we shall write $\Sigma \equiv \Sigma|_{\varepsilon=0}$, $\Gamma \equiv \Gamma|_{\varepsilon=0}$ and $\mathcal{S}(\Gamma) \equiv \mathcal{S}(\Gamma)|_{\varepsilon=0}$, while the linearized ST operator will be that defined in (4.1) by $\mathcal{B}_\Sigma|_{\varepsilon=0} = b_\Sigma$.

Let us consider

$$\mathcal{Q}' \equiv s + \omega\delta\,, \tag{5.1}$$

which, like $\mathcal{Q}$ in (3.1), describes a symmetry of $\mathcal{S}_{TYM}$, and it is nilpotent on shell.

Consider the composite operators

$$\Omega_{\phi^2} \equiv \mathrm{Tr}\,\omega^4\phi^2 \tag{5.2}$$

$$\Omega_{c^3} \equiv \mathrm{Tr}\left(-\omega^2 c\phi + \frac{1}{3}c^3\right). \tag{5.3}$$

The following relation holds

$$\Omega_{\phi^2} = \mathcal{Q}'\Omega_{c^3}\,. \tag{5.4}$$

The action

$$\Sigma' = \Sigma + S_{\sigma\lambda}\,, \tag{5.5}$$

where

$$S_{\sigma\lambda} = \int d^4x(\sigma\Omega_{\phi^2} + \lambda\Omega_{c^3})\,, \tag{5.6}$$

$(\sigma(x), \lambda(x))$ being external sources, satisfies the ST identity

$$\mathcal{S}'(\Sigma') = 0\,, \tag{5.7}$$

where

$$\mathcal{S}'(\Sigma') = \mathcal{S}(\Sigma') + \int d^4x\; \lambda\frac{\delta\Sigma'}{\delta\sigma}\,. \tag{5.8}$$

It is apparent from the expression (5.8) that the external fields $\sigma(x)$ and $\lambda(x)$, transforming one into the other, form a BRS doublet. This means that the cohomology of the linearized ST operator $b_{\Sigma'}$ does not depend on them [12].

In addition, the action Σ' satisfies the constraint

$$\int d^4x \left(\frac{\delta \Sigma'}{\delta c} + [\bar{c}, \frac{\delta \Sigma'}{\delta b}] + \lambda \frac{\delta \Sigma'}{\delta L} \right) = \Delta^{cl} , \tag{5.9}$$

where

$$\Delta^{cl} = \int d^4x \left([c, L] + [D, \phi] + [\Omega^\mu . A_\mu] + [\xi^\mu, \psi_\mu] + [\rho, \overline{\phi}] + [\tau, \eta] + \frac{1}{2}[B^{\mu\nu}, \chi_{\mu\nu}] \right) \tag{5.10}$$

is a classical breaking. Eq. (5.9) is known as the "ghost equation", and is a common feature of all gauge field theories in the Landau gauge [24].

Since the ordinary ST identity (3.16) is not affected by anomalies, and $(\sigma(x)\lambda(x))$ enter as a BRS doublet, the classical relation (5.7) can be extended at the quantum level, and we can write

$$\mathcal{S}'(\Gamma') = 0 , \tag{5.11}$$

where Γ' is the 1PI generating functional $\Gamma' = \Sigma' + O(\hbar)$.

Similarly, it is easy to prove that the ghost equation (5.9) is free of anomalies, and it holds for Γ', too.

Differentiating the ST identity (5.11) with respect to the external source $\lambda(x)$ and setting $\sigma(x) = \lambda(x) = 0$, we obtain

$$\frac{\delta}{\delta \lambda(x)} \mathcal{S}'(\Gamma') \Big|_{\sigma=\lambda=0} = 0 , \tag{5.12}$$

that is

$$b_{\Gamma'} \frac{\delta \Gamma'}{\delta \lambda(x)} \Big|_{\sigma=\lambda=0} = \frac{\delta \Gamma'}{\delta \sigma(x)} \Big|_{\sigma=\lambda=0} , \tag{5.13}$$

which means that the relation (5.4) between the composite operators Ω_{ϕ^2} (5.2) and Ω_{c^3} (5.3) holds true also at the quantum level:

$$\mathrm{Tr}\, \phi^2(x) \cdot \Gamma = b_\Gamma \left[\mathrm{Tr} \left(-\frac{c\phi}{\omega^2} + \frac{c^3}{3\omega^4} \right) (x) \cdot \Gamma \right] . \tag{5.14}$$

As a consequence of the ghost equation (5.9), the quantum insertion $\mathrm{Tr}(-c\phi/\omega^2 + c^3/3\omega^4)(x) \cdot \Gamma$ has vanishing anomalous dimension. Indeed, in the Landau gauge the composite operators given by polynomials of the Faddeev-Popov ghost field $c(x)$ do not renormalize, whereas it is the differentiated ghost $\partial_\mu c(x)$ which has quantum relevance [12, 24]:

$$\mathcal{C} \left[\mathrm{Tr} \left(-\frac{c\phi}{\omega^2} + \frac{c^3}{3\omega^4} \right) (x) \cdot \Gamma \right] = 0 , \tag{5.15}$$

JHEP05(2000)039

where $\mathcal{C}$ is the Callan-Symanzik (CS) operator

$$\mathcal{C} \equiv \mu\frac{\partial}{\partial\mu} + \hbar\beta_g\frac{\partial}{\partial g} - \hbar\sum_{\varphi}\gamma_{\varphi}\mathcal{N}_{\varphi}\,, \tag{5.16}$$

and

$$\mathcal{C}\Gamma = 0\,. \tag{5.17}$$

In (5.16) β_g is the β-function of the unique coupling constant g, γ_φ is the anomalous dimension of the generic field φ, the operator $\mathcal{N}_\varphi$ is the counting operator

$$\mathcal{N}_{\varphi} = \int d^4x\;\varphi\frac{\delta}{\delta\varphi}\,, \tag{5.18}$$

and μ is the renormalization scale which appears in the g-normalization condition, for instance that which fixes the transverse part of the two-point function

$$\frac{d}{dp^2}\Gamma^T(p^2)\bigg|_{p^2=-\mu^2} = -\frac{1}{g^2}\,. \tag{5.19}$$

The Callan-Symanzik operator $\mathcal{C}$ (5.16) commutes with the linearized ST operator b_Γ

$$[\mathcal{C}, b_\Gamma] = 0\,, \tag{5.20}$$

hence, by applying b_Γ to (5.15), we have

$$\mathcal{C}\left[\mathrm{Tr}\,\phi^2(x)\cdot\Gamma\right] = 0\,, \tag{5.21}$$

which completes the proof of vanishing anomalous dimension for the quantum insertion $\mathrm{Tr}\,\phi^2(x)\cdot\Gamma$.

It is also immediate to verify that the integrated quantum insertions

$$\int d^4x\mathcal{W}^4\,\mathrm{Tr}\,\phi^2\cdot\Gamma \tag{5.22}$$

and

$$\int d^4x\mathcal{W}^4\,\mathrm{Tr}\left(-\frac{c\phi}{\omega^2}+\frac{c^3}{3\omega^4}\right)\cdot\Gamma \tag{5.23}$$

have vanishing dimensions too, i.e.

$$\mathcal{C}\int d^4x\mathcal{W}^4\,\mathrm{Tr}\,\phi^2\cdot\Gamma = 0 \tag{5.24}$$

$$\mathcal{C}\int d^4x\mathcal{W}^4\,\mathrm{Tr}\left(-\frac{c\phi}{\omega^2}+\frac{c^3}{3\omega^4}\right)\cdot\Gamma = 0\,. \tag{5.25}$$

In fact, the same procedure used to show (5.21) can be repeated, by coupling two global parameters to the integrated composite operators (5.22) and (5.23). By using the ghost equation (5.9), which holds unaltered, and the fact that, on integrated functionals, the operators b_Γ and $\mathcal{W}_\mu$ commute (4.2), one can play with the CS operator and reach the desired results (5.24) and (5.25).

6. The non-renormalization theorem for the beta function

The starting point is the classical relation, deriving from the expression of the twisted $N = 2$ SYM action (4.6), at $\varepsilon^\mu \equiv 0$

$$\frac{\partial \Sigma}{\partial g} = \frac{2}{3g^3} \int d^4x \mathcal{W}^4 \, \mathrm{Tr} \, \frac{\phi^2}{2} + b_\Sigma \int d^4x \, \mathrm{Tr}(\phi D - \xi^\mu \psi_\mu) \, . \tag{6.1}$$

The above expression can be written

$$\frac{\partial \Sigma}{\partial g} = \frac{1}{g^3} b_\Sigma \, \Xi \, , \tag{6.2}$$

with Ξ given by

$$\Xi = \frac{2}{3} \int d^4x \mathcal{W}^4 \, \mathrm{Tr} \left(-\frac{c\phi}{\omega^2} + \frac{c^3}{\omega^4} \right) + g^3 \int d^4x \, \mathrm{Tr}(\phi D - \xi^\mu \psi_\mu) \, . \tag{6.3}$$

Our first intermediate task is the quantum extension of (6.2). In order to do this, consider the action, modified by the introduction of a parameter α

$$\Sigma' = \Sigma + \frac{\alpha}{g^3} \Xi \, , \tag{6.4}$$

which is easily seen to satisfy the identity

$$\mathcal{S}(\Sigma') + \alpha \frac{\partial \Sigma'}{\partial g} = 0 \, . \tag{6.5}$$

The fact that α and g transform as a BRS doublet, allows us to implement (6.5) at the quantum level

$$\mathcal{S}(\Gamma') + \alpha \frac{\partial \Gamma'}{\partial g} = 0 \, . \tag{6.6}$$

Deriving (6.6) with respect to α and setting $\alpha = 0$, we find

$$b_{\Gamma'} \left. \frac{\partial \Gamma'}{\partial \alpha} \right|_{\alpha=0} = \left. \frac{\partial \Gamma'}{\partial g} \right|_{\alpha=0} \, . \tag{6.7}$$

Now, at the classical level, from (6.4) we have

$$\frac{\partial \Sigma'}{\partial \alpha} = \frac{1}{g^3} \Xi \, , \tag{6.8}$$

and Ξ consists of two terms: $\int d^4x \mathcal{W}^4 \, \mathrm{Tr}(-c\phi/\omega^2 + c^3/\omega^4)$, which has vanishing anomalous dimension, due to the nonrenormalization theorem (5.25); and $g^3 \int d^4x \, \mathrm{Tr}(\phi D - \xi^\mu \psi_\mu)$, which is purely classic, being a functional linear in the quantum fields $\phi(x)$ and $\psi_\mu(x)$.

Consequently, we can extend (6.8) to the quantum level

$$\frac{\partial \Gamma'}{\partial \alpha} = \frac{1}{g^3} \Xi \cdot \Gamma' \tag{6.9}$$

and, from the relation (6.7), we have

$$\frac{\partial \Gamma}{\partial g} = \frac{1}{g^3} b_\Gamma (\Xi \cdot \Gamma) \,, \tag{6.10}$$

or

$$\frac{\partial \Gamma}{\partial g} = \frac{2}{3g^3} \int d^4x \left(\mathcal{W}^4 \operatorname{Tr} \phi^2 \right) \cdot \Gamma + b_\Gamma \int d^4x \operatorname{Tr}(\phi D - \xi^\mu \psi_\mu) \cdot \Gamma \,, \tag{6.11}$$

which realizes our intermediate aim.

All the tools to prove the 1-loop finiteness of the beta function of the coupling constant are now at our disposal.

The 1PI generating functional of the theory Γ satisfies the CS equation (5.17), or explicitly

$$\mu \frac{\partial \Gamma}{\partial \mu} + \hbar \beta_g \frac{\partial \Gamma}{\partial g} - \hbar \sum_\varphi \gamma_\varphi b_\Gamma (\Omega \cdot \Gamma) = 0 \,, \tag{6.12}$$

where we used the fact that the anomalous dimensions belong to the BRS-trivial sector of the counterterms. Deriving (6.12) with respect to the coupling constant g we have

$$\mu \frac{\partial}{\partial \mu} \frac{\partial \Gamma}{\partial g} + \hbar \frac{\partial \beta_g}{\partial g} \frac{\partial \Gamma}{\partial g} + \hbar \beta_g \frac{\partial}{\partial g} \frac{\partial \Gamma}{\partial g} - \hbar b_\Gamma (\Omega' \cdot \Gamma) = 0 \,, \tag{6.13}$$

with

$$\Omega' \cdot \Gamma \equiv \sum_\varphi \left(\frac{\partial \gamma_\varphi}{\partial g} \Omega \cdot \Gamma + \gamma_\varphi \frac{\partial}{\partial g} \Omega \cdot \Gamma \right) . \tag{6.14}$$

It is important to emphasize that the quantum insertion $\Omega \cdot \Gamma$, and hence $\Omega' \cdot \Gamma$, coming from the expression of counterterms, are analytic in the ω-parameter, as already remarked in section 4.

Substituting in (6.13) the expression obtained in (6.11) for the quantum insertion $\partial \Gamma / \partial g$, and collecting all the exact terms at the r.h.s., we find

$$\mathcal{C} \left[\frac{1}{3g^3} \int d^4x \left(\mathcal{W}^4 \operatorname{Tr} \Phi^2 \right) \cdot \Gamma \right] + \hbar \frac{\partial \beta_g}{\partial g} \left[\frac{1}{3g^3} \int d^4x \left(\mathcal{W}^4 \operatorname{Tr} \Phi^2 \right) \cdot \Gamma \right] = \hbar b_\Gamma (\Omega'' \cdot \Gamma) \,, \tag{6.15}$$

where $\Omega'' \cdot \Gamma$ is again analytic in the parameter ω, being directly related to the expression of the classical action and of the counterterms. Remembering now the result (5.24), stating that the quantum insertion $\int d^4x \left(\mathcal{W}^4 \operatorname{Tr} \phi^2 \right) \cdot \Gamma$ has vanishing anomalous dimension, eq. (6.15) becomes

$$\left(-\frac{3}{g^4} \beta_g + \frac{1}{g^3} \frac{\partial \beta_g}{\partial g} \right) \frac{1}{3} \int d^4x \left(\mathcal{W}^4 \operatorname{Tr} \phi^2 \right) \cdot \Gamma = b_\Gamma (\Omega'' \cdot \Gamma) \,. \tag{6.16}$$

Both sides of (6.16) can be written as cocycles, since it is evident that

$$\int d^4x \left(\mathcal{W}^4 \operatorname{Tr}\phi^2\right)\cdot\Gamma = b_\Gamma \int d^4x \left[\mathcal{W}^4 \operatorname{Tr}\left(-\frac{c\phi}{\omega^2}+\frac{c^3}{3\omega^4}\right)\right]\cdot\Gamma\,, \tag{6.17}$$

which derives immediately from (5.14).

We thus have

$$\left(-\frac{3}{g^4}\beta_g+\frac{1}{g^3}\frac{\partial\beta_g}{\partial g}\right) b_\Gamma \int d^4x \left[\mathcal{W}^4 \operatorname{Tr}\left(-\frac{c\phi}{\omega^2}+\frac{c^3}{3\omega^4}\right)\right]\cdot\Gamma = b_\Gamma \sum_k \omega^k(\Omega''_{(k)}\cdot\Gamma)\,. \tag{6.18}$$

That is, we have an equality between an analytic function of ω, in the r.h.s., and a non-analytic function on the l.h.s. The only solution is that the function β_g satisfies the differential equation

$$\frac{3}{g}\beta_g+\partial_g\beta_g=0\,, \tag{6.19}$$

i.e.

$$\beta_g = Kg^3 \qquad (K\text{ constant}). \tag{6.20}$$

Eq. (6.20) is our result. It states that, to all orders of perturbation theory, the β-function of the coupling constant receives contributions only to the 1-loop order.

The exact statement is that a renormalization scheme does exist such that eq. (6.20) is true, namely that all higher corrections vanish. Such scheme can be identified with the requirement that the insertion $\partial\Gamma/\partial g$ can be defined exactly as expressed in eq. (6.11), which is of the type

$$\frac{\partial\Gamma}{\partial g}=\frac{1}{g^3}\int d^4x\Omega\cdot\Gamma + b_\Gamma(\Delta\cdot\Gamma)\,. \tag{6.21}$$

7. Summary

In this paper we gave an algebraic, regularization independent, proof of the non-renormalization theorem concerning the vanishing above one loop of the perturbative β-function of $N=2$ SYM theory. The key points of our approach have been the followings:

The twist: by means of the twisting procedure, we passed from $N=2$ Super Yang-Mills theory to the equivalent Topological Yang-Mills theory formulated on $\mathbb{R}^4$.

$\operatorname{Tr}\phi^2$**:** we were able to write the classical action of the twisted theory as (4.6)

$$\Sigma \approx -\frac{1}{3g^2}\mathcal{W}^4\int d^4x \operatorname{Tr}\frac{\phi^2}{2}\,,$$

modulo a trivial BRS cocycle; i.e. we related the classical action to $\operatorname{Tr}\phi^2$, which is a gauge invariant polynomial in the scalar field $\phi(x)$ of the (untwisted) $N=2$ gauge supermultiplet.

JHEP05(2000)039

Nonrenormalization of $\mathrm{Tr}\,\phi^2$**:** we have proved that $\mathrm{Tr}\,\phi^2$ has vanishing anomalous dimension.

Callan-Symanzik equation: a wise manipulation of the Callan-Symanzik equation implied that the beta function of the unique coupling constant of the theory satisfies, to all orders of perturbation theory, the following differential equation (6.19)

$$\frac{3}{g}\beta_g + \partial_g \beta_g = 0\,,$$

which yields the result (6.20)

$$\beta_g = K g^3\,,$$

i.e. the β function has only one loop contribution, in the scheme defined by (6.21).

Acknowledgments

It is a pleasure to thank Carlo Maria Becchi, Beppe Bandelloni and Nico Magnoli for discussions and for their comments. A.B., N.M. and A.T. would like to thank UERJ, where most of this work has been done, and the people of CBPF (Rio de Janeiro) for the warm hospitality and the friendly athmosphere. The financial support of the FAPERJ, Fundação de Amparo a Pesquisa do Estado do Rio de Janeiro, and the SR2-UERJ are gratefully acknowledged.

References

[1] For a recent review, cfr. M. Shifman and A. Vainshtein, *Instantons versus supersymmetry: fifteen years later*, in *ITEP lectures in particle physics and field theory*, M. Shifman ed., World Scientific, Singapore 1999, vol. 2, pp. 485–648 [hep-th/9902018].

[2] D. Amati, K. Konishi, Y. Meurice, G.C. Rossi and G. Veneziano, *Nonperturbative aspects in supersymmetric gauge theories*, *Phys. Rep.* **162** (1988) 169.

[3] N. Seiberg and E. Witten, *Electric-magnetic duality, monopole condensation and confinement in $N = 2$ supersymmetric Yang-Mills theory*, *Nucl. Phys.* **B 426** (1994) 19 [hep-th/9407087].

[4] P. West, *Introduction to supersymmetry and supergravity*, World Scientific, Singapore 1990.

[5] E. Witten, *Topological quantum field theory*, *Comm. Math. Phys.* **117** (1988) 353.

[6] F. Fucito, A. Tanzini, L.C. Vilar, O.S. Ventura, C.A. Sasaki and S.P. Sorella, *Algebraic renormalization: perturbative twisted considerations on topological Yang-Mills theory and on $N = 2$ supersymmetric gauge theories*, lectures given at the *First school*

on field theory and gravitation, Vitória, Espírito Santo, Brazil, hep-th/9707209;
A. Tanzini, O.S. Ventura, L.C.Q. Vilar and S.P. Sorella, *BRST cohomology of $N=2$ super Yang-Mills theory in 4D*, *J. Phys.* **G 26** (2000) 1 [hep-th/9811191].

[7] D. Bellisai, F. Fucito, A. Tanzini and G. Travaglini, *Instanton calculus, topological field theories and $N=2$ super Yang-Mills theories*, hep-th/0003272;
Multi-instantons, supersymmetry and topological field theories, *Phys. Lett.* **B 480** (2000) 365 [hep-th/0002110].

[8] N. Seiberg, *Supersymmetry and nonperturbative beta functions*, *Phys. Lett.* **B 206** (1988) 75.

[9] M. Marino, *The geometry of supersymmetric gauge theories in four dimensions*, Ph.D. thesis, hep-th/9701128.

[10] J. Bagger and J. Wess, *Supersymmetry and supergravity*, Princeton University Press.

[11] F. Delduc, F. Gieres and S.P. Sorella, *Supersymmetry of the $D=3$ Chern-Simons action in the Landau gauge*, *Phys. Lett.* **B 225** (1989) 367;
N. Maggiore and S.P. Sorella, *Finiteness of the topological models in the landau gauge*, *Nucl. Phys.* **B 377** (1992) 236.

[12] O. Piguet and S.P. Sorella, *Algebraic renormalization: Perturbative renormalization, symmetries and anomalies*, Springer, Berlin 1995.

[13] P. Breitenlohner and D. Maison, *Renormalization of supersymmetric Yang-Mills theories*, in *Cambridge 1985, Proceedings, supersymmetry and its applications*, p. 309.

[14] N. Maggiore, *Algebraic renormalization of $N=2$ super Yang-Mills theories coupled to matter*, *Int. J. Mod. Phys.* **A 10** (1995) 3781 [hep-th/9501057].

[15] R. Brooks, D. Montano and J. Sonnenschein, *Gauge fixing and renormalization in topological quantum field theory*, *Phys. Lett.* **B 214** (1988) 91.

[16] M.F. Sohnius and P.C. West, *Conformal invariance in $N=4$ supersymmetric Yang-Mills theory*, *Phys. Lett.* **B 100** (1981) 245.

[17] P.S. Howe, K.S. Stelle and P.C. West, *A class of finite four-dimensional supersymmetric field theories*, *Phys. Lett.* **B 124** (1983) 55.

[18] D.J. Gross, *Applications of the renormalization group to high-energy physics*, in *Les Houches 1975, Proceedings, methods in field theory*, Amsterdam 1976, p. 141.

[19] P.L. White, *An analysis of the cohomology structure of super Yang-Mills coupled to matter*, *Class. and Quant. Grav.* **9** (1992) 1663.

[20] N. Maggiore, *Off-shell formulation of $N=2$ super Yang-Mills theories coupled to matter without auxiliary fields*, *Int. J. Mod. Phys.* **A 10** (1995) 3937 [hep-th/9412092].

[21] J.A. Dixon, *Calculation of BRS cohomology with spectral sequences*, *Comm. Math. Phys.* **139** (1991) 495.

[22] F. Delduc, N. Maggiore, O. Piguet and S. Wolf, *Note on constrained cohomology*, *Phys. Lett.* **B 385** (1996) 132 [hep-th/9605158].

[23] S. Ouvry, R. Stora and P. van Baal, *On the algebraic characterization of Witten's topological Yang-Mills theory*, *Phys. Lett.* **B 220** (1989) 159;
J. Kalkman, *Brst model for equivariant cohomology and representatives for the equivariant Thom class*, *Comm. Math. Phys.* **153** (1993) 447;
R. Stora, *Equivariant cohomology and topological theories*, in *BRS symmetry*, M. Abe, N. Nakanishi and Iojima eds., Universal Academy Press, Tokyo, Japan, 1996;
R. Stora, F. Thuillier and J.C. Wallet, *Algebraic structure of cohomological field theory models and equivariant cohomology*, lectures given at the *First caribbean spring school of mathematics and theoretical physics*, R. Coquereaux, M. Dubois-Violette and P. Flad Eds., World Scientific Publ., 1995;
R. Stora, *Exercises in equivariant cohomology*, hep-th/9611114; *On gauge fixing considered as a fine art and on the Slavnov symmetry that follows*, (in french), hep-th/9611115; *Exercises in equivariant cohomology and topological theories*, hep-th/9611116.

[24] A. Blasi, O. Piguet and S.P. Sorella, *Landau gauge and finiteness*, *Nucl. Phys.* **B 356** (1991) 154.